

Some fixed point theorems in complex valued b -metric spaces

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Abstract. We introduce the notion of fixed points for a mappings in complex valued b -metric space and demonstrate the existence and uniqueness of the main Banach contractive type, Kannan type, and Chatterjea type in complex valued b -metric spaces. Presented theorems in this paper extend and generalize the results derived by Mehmet and Kiziltunc in [12]. Some examples are given to illustrate the main results.

Keywords: b -metric spaces, complex valued, b -metric spaces, fixed point.

1. Introduction

Banach [9] introduced the main contraction principle for the fixed point theory which was the starting point for many researchers in nonlinear analysis. In 1989, Bakhtin [8] introduced the concept of a b -metric space as a generalization of metric spaces. In 1993, Czerwik [10, 11] extended many results related to the b -metric spaces. Close to our interest in this paper many researchers studied some fixed point theorems in the so called b -metric space [1, 2, 3, 4, 5, 12, 13, 14]. In 2011, Azam et al. [17] introduced the concept of complex valued metric spaces and obtained sufficient conditions for the existence of common fixed points of a pair of contractive type mappings. The existence and uniqueness of the fixed points of self-mappings noted were considered in [18, 19]. In 2013, Rao et al [20] introduced the notion of complex valued b -metric spaces which is more general than the well known complex valued metric spaces and also gave common fixed point theorems.

In this paper, we generalize results of Mehmet and Kiziltunc in [12], by introducing the contractive mapping in a complete complex valued b -metric space.

Definition 1.1 ([7]). *Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \rightarrow [0, \infty)$ is called a b -metric if for all $x, y, z \in X$ the following conditions are satisfied:*

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

$$(iii) \quad d(x, y) \leq s[d(x, z) + d(z, y)].$$

The pair (X, d) is called a b -metric space. The number $s \geq 1$ is called the coefficient of (X, d) .

Example 1.1 ([6]). Let $X = \{x_1, x_2, x_3, x_4\}$. Define a function $d : X \times X \rightarrow [0, \infty)$ such that $d(x_1, x_2) = k \geq 2$ and $d(x_1, x_3) = d(x_1, x_4) = d(x_2, x_3) = d(x_2, x_4) = d(x_3, x_4) = 1$, $d(x_i, x_j) = d(x_j, x_i)$ for all $i, j = 1, 2, 3, 4$ and $d(x_i, x_i) = 0$ for all $i, j = 1, 2, 3, 4$. Then $d(x_i, x_j) \leq \frac{k}{2}[d(x_i, x_n) + d(x_n, x_j)]$ for $x_i \in X$ and $i, j = 1, 2, 3, 4$. Therefore, (X, d) is a b -metric space with constant $s = \frac{k}{2}$. However if $k > 2$ the ordinary triangle inequality does not hold and thus (X, d) is not a metric space.

Let \mathbb{C} be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order \preceq on \mathbb{C} as follows:

$$z_1 \preceq z_2 \text{ if and only if } \operatorname{Re}(z_1) \leq \operatorname{Re}(z_2), \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2)$$

Thus $z_1 \preceq z_2$ if one of the following holds:

- (1) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,
- (2) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,
- (3) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,
- (4) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$.

We will write $z_1 \prec z_2$ if $z_1 \neq z_2$ and one of (2),(3) and (4) is satisfied, also we will write $z_1 \prec z_2$ if only (4) is satisfied.

Remark 1.1. We can easily check that the following statements are hold:

- (i) If $a, b \in \mathbb{R}$ and $a \leq b$, then $az \preceq bz$ for all $z \in \mathbb{C}$,
- (ii) If $0 \preceq z_1 \prec z_2$, then $|z_1| < |z_2|$,
- (iii) If $z_1 \preceq z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 1.2 ([17]). Let X be a nonempty set. A function $d : X \times X \rightarrow \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

- (i) $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$;
- (iii) $d(x, y) \preceq d(x, z) + d(z, y)$.

The pair (X, d) is called a complex valued metric space.

Example 1.2 ([18]). Let $X = \mathbb{C}$. Define the mapping $d : X \times X \rightarrow \mathbb{C}$ by $d(x, y) = i|x - y|$, for all $x, y \in X$. Then (X, d) is a complex valued metric space.

Example 1.3 ([19]). Let $X = \mathbb{C}$. Define the mapping $d : X \times X \rightarrow \mathbb{C}$ by $d(x, y) = e^{ik}|x - y|$, where $k \in \mathbb{R}$ and for all $x, y \in X$. Then (X, d) is a complex valued metric space.

Definition 1.3 ([20]). Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \rightarrow \mathbb{C}$ is called a complex valued b -metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

- (i) $0 \lesssim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$;
- (iii) $d(x, y) \lesssim s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a complex valued b -metric space.

Example 1.4 ([20]). Let $X = [0, 1]$. Define the mapping $d : X \times X \rightarrow \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$, for all $x, y \in X$. Then (X, d) is a complex valued b -metric space with $s = 2$.

Definition 1.4 ([20]). Let (X, d) be a complex valued b -metric space.

- (i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) := \{y \in X : d(x, y) \prec r\} \subseteq A$.
- (ii) A point $x \in X$ is called a limit point of a set A whenever for every $0 \prec r \in \mathbb{C}$, $B(x, r) \cap (A - X) \neq \emptyset$.
- (iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A .
- (iv) A subset $A \subseteq X$ is called closed whenever each element of A belongs to A .
- (v) A sub-basis for a Hausdorff topology τ on X is a family $F = \{B(x, r) : x \in X \text{ and } 0 \prec r\}$.

Definition 1.5 ([20]). Let (X, d) be a complex valued b -metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

- (i) If for every $c \in \mathbb{C}$, with $0 \prec r$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $\{x_n\} \rightarrow x$ as $n \rightarrow \infty$.
- (ii) If for every $c \in \mathbb{C}$, with $0 \prec r$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.
- (iii) If every Cauchy sequence in X is convergent, then (X, d) is said to be a complete complex valued b -metric space.

Lemma 1.1 ([20]). *Let (X, d) be a complex valued b -metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \rightarrow 0$ as $n \rightarrow \infty$.*

Lemma 1.2 ([20]). *Let (X, d) be a complex valued b -metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \rightarrow 0$ as $n \rightarrow \infty$, where $m \in \mathbb{N}$.*

2. Main result

Definition 2.1. *Let (X, d) be a complex valued b -metric space with the coefficient $s \geq 1$. A mappings $T : X \rightarrow X$ is called contraction if there exists a constant $k \in [0, 1)$*

$$(2.1) \quad d(Tx, Ty) \lesssim kd(x, y)$$

for all $x, y \in X$.

Definition 2.2. *Let (X, d) be a complex valued b -metric space with the coefficient $s \geq 1$. A mapping $T : X \rightarrow X$ is called weak contraction if there exists a constant $\alpha \in (0, 1)$ and $\beta \geq 0$ such that*

$$(2.2) \quad d(Tx, Ty) \lesssim \alpha d(x, y) + \beta d(y, Tx)$$

for all $x, y \in X$.

Remark 2.1. Note that by using the symmetry of the distance, the weak contraction condition (2.2) implies

$$(2.3) \quad d(Tx, Ty) \lesssim \alpha d(x, y) + \beta d(x, Ty)$$

for all $x, y \in X$. So, we have to verify (2.2) and (2.3) in order to check the weak contractiveness of T .

Remark 2.2. It is clear that any contraction mapping is also weak contraction in the (usual) complex b -metric space.

Our next theorem is about the generalization of the main Banach's contraction theorem in complex valued b -metric spaces.

Theorem 2.1. *Let (X, d) be a complete complex valued b -metric space with the coefficient $s \geq 1$ and $T : X \rightarrow X$ be a mapping satisfying:*

$$(2.4) \quad d(Tx, Ty) \lesssim kd(x, y)$$

where $k \in [0, 1)$ and $sk < 1$. Then, T has a unique fixed point in X .

Proof. For any arbitrary point $x_0 \in X$. Define sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n = T^n x_0$. Then $d(x_1, x_2) = d(Tx_0, Tx_1) \lesssim kd(x_0, x_1)$, and also $d(x_2, x_3) = d(Tx_1, Tx_2) \lesssim kd(x_1, x_2) \lesssim k^2d(x_0, x_1)$. By induction on n we get

$$(2.5) \quad d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \lesssim kd(x_{n-1}, x_n) \lesssim k^n d(x_0, x_1).$$

Since $k \in [0, 1)$ we get $k^n \in [0, 1)$. Therefore, T is a contraction mapping. Now we show that $\{x_n\}$ is a Cauchy sequence in (X, d) . Let $m > n > 0$. Thus,

$$\begin{aligned} d(x_n, x_m) &\lesssim sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\lesssim sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^2d(x_{n+2}, x_m) \\ &\lesssim sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^3d(x_{n+2}, x_m) + s^3d(x_{n+2}, x_m) \\ &\lesssim sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^3d(x_{n+2}, x_m) \\ &+ \dots + s^{m-n-2}d(x_{m-3}, x_{m-2}) + s^{m-n-1}d(x_{m-2}, x_{m-1}) \\ &+ s^{m-n-1}d(x_{m-1}, x_m) \\ &\lesssim sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^3d(x_{n+2}, x_m) \\ &+ \dots + s^{m-n-2}d(x_{m-3}, x_{m-2}) + s^{m-n-1}d(x_{m-2}, x_{m-1}) \\ &+ s^{m-n}d(x_{m-1}, x_m). \end{aligned}$$

By using (2.5) we get

$$\begin{aligned} d(x_n, x_m) &\lesssim sk^n d(x_0, x_1) + s^2k^{n+1}d(x_0, x_1) + s^3k^{n+2}d(x_0, x_1) \\ &+ \dots + s^{m-n-2}k^{m-3}d(x_0, x_1) + s^{m-n-1}k^{m-2}d(x_0, x_1) \\ &+ s^{m-n}k^{m-1}d(x_0, x_1) = \sum_{i=1}^{m-n} s^i k^{i+n-1} d(x_0, x_1) \end{aligned}$$

Therefore,

$$\begin{aligned} d(x_n, x_m) &\lesssim \sum_{i=1}^{m-n} s^{i+n-1} k^{i+n-1} d(x_0, x_1) \\ &= \sum_{t=n}^{m-1} s^t k^t d(x_0, x_1) \lesssim \sum_{t=n}^{\infty} (sk)^t d(x_0, x_1) = \frac{(sk)^n}{1 - sk} d(x_0, x_1). \end{aligned}$$

Taking the modulus, we get

$$(2.6) \quad |d(x_n, x_m)| \leq \frac{(sk)^n}{1 - sk} |d(x_0, x_1)| \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

Therefore

$$(2.7) \quad \lim_{n, m \rightarrow \infty} |d(x_n, x_m)| = 0.$$

Hence, $\{x_n\}$ is a Cauchy sequence in X .

Since X is complete, there exists some $x \in X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Now we show that $Tx = x$. Assume not, then there exist $z \in X$ such that

$$(2.8) \quad |d(Tx, x)| = |z| > 0.$$

So, we have

$$\begin{aligned} z &= d(x, Tx) \lesssim sd(x, x_n) + sd(x_n, Tx) = sd(x, x_n) + sd(Tx_{n-1}, Tx) \\ &\lesssim sd(x, x_n) + skd(x_{n-1}, x). \end{aligned}$$

This implies that

$$(2.9) \quad |z| = |d(x, Tx)| \leq s|d(x, x_n)| + sk|d(x_{n-1}, x)| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

we obtain that $|z| = |d(x, Tx)| \leq 0$, a contradiction with (2.8). So $|z| = |d(x, Tx)| = 0$, Hence $Tx = x$.

Now we show that T has a unique fixed point. For this, assume that x^* is another fixed point of T . Then by using (2.1) we get

$$d(x, x^*) = d(Tx, Tx^*) \lesssim kd(x, x^*) \prec d(x, x^*).$$

This implies that $|d(x, x^*)| < |d(x, x^*)|$, a contradiction. Thus x is a unique fixed point in X . This completes the proof. \square

Remark 2.3. Any mapping satisfying the contractive condition in (2.1) need not be a contraction in complex valued b -metric spaces unless some conditions on k .

In the following theorem we extend the Kannan type fixed point theorem in [15] in complex valued b -metric spaces.

Theorem 2.2. *Let (X, d) be a complete complex valued b -metric space with the coefficient $s \geq 1$ and $T : X \rightarrow X$ be a mapping satisfying:*

$$(2.10) \quad d(Tx, Ty) \lesssim \mu[d(x, Tx) + d(y, Ty)]$$

where $\mu \in [0, \frac{1}{2})$. Then, T has a unique fixed point in X .

Proof. For any arbitrary point $x_0 \in X$. Define sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n = T^n x_0$. Then

$$d(x_1, x_2) = d(Tx_0, Tx_1) \lesssim \mu[d(x_0, Tx_0) + d(x_1, Tx_1)],$$

and also

$$d(x_2, x_3) = d(Tx_1, Tx_2) \lesssim \mu[d(x_1, Tx_1) + d(x_2, Tx_2)].$$

By induction on n we get $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \lesssim \mu[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] = \mu[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$. Thus we obtain

$$(2.11) \quad d(x_n, x_{n+1}) \lesssim \frac{\mu}{1-\mu} d(x_{n-1}, x_n).$$

Therefore

$$\begin{aligned} d(x_n, x_{n+1}) &\lesssim \frac{\mu}{1-\mu} d(x_{n-1}, x_n) \\ &\lesssim \left(\frac{\mu}{1-\mu}\right)^2 d(x_{n-2}, x_{n-1}) \\ &\lesssim \left(\frac{\mu}{1-\mu}\right)^3 d(x_{n-3}, x_{n-2}) \\ &\lesssim \dots \lesssim \left(\frac{\mu}{1-\mu}\right)^n d(x_0, x_1). \end{aligned}$$

Since $\mu \in [0, \frac{1}{2})$, we get $(\frac{\mu}{1-\mu})^n \in [0, 1)$. Therefore, T is a contraction mapping. Easily as in the same manner in Theorem 2.1 we can show that $\{x_n\}$ is a Cauchy sequence in (X, d) . Since X is complete, there exists some $x \in X$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Now we show that $Tx = x$. Assume not, then there exist $z \in X$ such that

$$(2.12) \quad |d(Tx, x)| = |z| > 0.$$

So, we have

$$\begin{aligned} z &= d(x, Tx) \lesssim sd(x, x_n) + sd(x_n, Tx) = sd(x, x_n) + sd(Tx_{n-1}, Tx) \\ &\lesssim sd(x, x_n) + s\mu[d(x_{n-1}, Tx_{n-1}) + d(x, Tx)] \\ &= sd(x, x_n) + s\mu d(x_{n-1}, x_n) + s\mu d(x, Tx). \end{aligned}$$

This implies that

$$(2.13) \quad \begin{aligned} d(x, Tx) &\lesssim \frac{s}{1-s\mu} d(x, x_n) + \frac{s\mu}{1-s\mu} d(x_{n-1}, x_n) \\ &\lesssim \frac{s}{1-s\mu} d(x, x_n) + \frac{s\mu}{1-s\mu} \left(\frac{\mu}{1-\mu}\right)^{n-1} d(x_0, x_1). \end{aligned}$$

Taking modulus in (2.13) we get

$$(2.14) \quad \begin{aligned} |z| &= |d(x, Tx)| \\ &\leq \frac{s}{1-s\mu} |d(x, x_n)| + \frac{s\mu}{1-s\mu} \left(\frac{\mu}{1-\mu}\right)^{n-1} |d(x_0, x_1)| \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

we obtain that $|z| = |d(x, Tx)| \leq 0$, a contradiction with (2.12). So $|z| = |d(x, Tx)| = 0$, Hence $Tx = x$. As in Theorem 2.1 it is easily to show that T has a unique fixed point. This completes the proof. \square

Remark 2.4. It is clear that we have no restriction on the contraction mapping (2.10) for applying the Kannan's fixed point theorem complex valued b -metric spaces.

Remark 2.5 (see also [12]). Any mapping satisfying the contractive condition (2.10) need not be a weak contraction in complex valued b -metric spaces unless $s\mu \in [0, \frac{1}{2})$

In the following theorem we extend the Chatterjea type fixed point theorem in [16] in complex valued b -metric spaces.

Theorem 2.3. *Let (X, d) be a complete complex valued b -metric space with the coefficient $s \geq 1$ and $T : X \rightarrow X$ be a mapping satisfying:*

$$(2.15) \quad d(Tx, Ty) \lesssim \lambda[d(x, Ty) + d(y, Tx)]$$

where $s\lambda \in [0, \frac{1}{2})$. Then, T has a unique fixed point in X .

Proof. For any arbitrary point $x_0 \in X$. Define sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n = T^n x_0$. Then

$$\begin{aligned} d(x_1, x_2) &= d(Tx_0, Tx_1) \lesssim \lambda[d(x_0, Tx_1) + d(x_1, Tx_0)] \\ &= \lambda[d(x_0, x_2) + d(x_1, x_1)] = \lambda d(x_0, x_2), \end{aligned}$$

and

$$\begin{aligned} d(x_2, x_3) &= d(Tx_1, Tx_2) \lesssim \lambda[d(x_1, Tx_2) + d(x_2, Tx_1)] \\ &= \lambda[d(x_1, x_3) + d(x_2, x_2)] = \lambda d(x_1, x_3), \end{aligned}$$

By induction on n we get

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\lesssim \lambda[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] \\ &= \lambda[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\ &= \lambda d(x_{n-1}, x_{n+1}) \\ &\lesssim s\lambda[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \end{aligned}$$

Thus we obtain

$$d(x_n, x_{n+1}) \lesssim \frac{s\lambda}{1-s\lambda} d(x_{n-1}, x_n).$$

Since $s\lambda \in [0, \frac{1}{2})$, we get $\frac{s\lambda}{1-s\lambda} \in [0, 1)$. Therefore, T is a contraction mapping. Easily as in the same manner in Theorem 2.1 and Theorem 2.2 we can show that $\{x_n\}$ is a Cauchy sequence in X and hence it is convergent. Since X is complete. Now we show that x is a fixed point of T . Assume not, then there exist $z \in X$ such that

$$(2.16) \quad |d(Tx, x)| = |z| > 0.$$

So, we have

$$\begin{aligned}
z &= d(x, Tx) \\
&\succcurlyeq sd(x, x_{n+1}) + sd(x_{n+1}, Tx) \\
&= sd(x, x_{n+1}) + sd(Tx_n, Tx) \\
&\succcurlyeq sd(x, x_{n+1}) + s\lambda[d(x_n, Tx) + d(x, Tx_n)] \\
&= sd(x, x_{n+1}) + s\lambda d(x_n, Tx) + s\lambda d(x, x_{n+1}).
\end{aligned}$$

Taking modulus in the above inequality we get

$$(2.17) \quad |z| = |d(x, Tx)| \leq s|d(x, x_{n+1})| + s\lambda|d(x_n, Tx)| + s\lambda|d(x, x_{n+1})|.$$

Taking the limit of (2.17) as $n \rightarrow \infty$, we get

$$(2.18) \quad |z| = |d(x, Tx)| \leq s\lambda|d(x, Tx)| < \frac{1}{2}|z|,$$

The inequality (2.18) true only if $|z| = |d(x, Tx)| = 0$, Hence $Tx = x$. Finally, we show that x is the unique fixed point of T . Assume not, then there exist x^* another fixed point of T . Then, by using (2.1) we get

$$\begin{aligned}
d(x, x^*) &= d(Tx, Tx^*) \\
&\succcurlyeq \lambda[d(x, Tx^*) + d(x^*, Tx)] \\
(2.19) \quad &= \lambda[d(x, x^*) + d(x^*, x)] = 2\lambda d(x, x^*).
\end{aligned}$$

Taking modulus of (2.19), and since $\lambda < \frac{1}{2}$ we get

$$|d(x, x^*)| \leq 2\lambda|d(x, x^*)| < |d(x, x^*)|$$

a contradiction. we deduce that T has a unique fixed point. This completes the proof. \square

Example 2.1. Let $X = \{x_1, x_2, x_3, x_4\}$. Define a function $d : X \times X \rightarrow \mathbb{C}$ such that

$$d(x, y) = \begin{cases} d(x_1, x_2) = 3i, \\ d(x_1, x_3) = d(x_2, x_3) = i, \\ d(x_1, x_4) = d(x_2, x_4) = d(x_3, x_4) = 5i, \\ d(x_m, x_n) = d(x_n, x_m), & \text{for all } m, n = 1, 2, 3, 4 \\ d(x_m, x_m) = 0, & \text{for all } m = 1, 2, 3, 4. \end{cases}$$

Then $d(x_m, x_n) \leq \frac{k}{2}[d(x_m, x_l) + d(x_l, x_n)]$ for $x_m, x_n, x_l \in X, k \geq 3$ and $m, n, l = 1, 2, 3, 4$. Therefore, (X, d) is a complex valued b -metric space with constant $s = \frac{3}{2}$.

Let $x = x_1, y = x_2$, and $z = x_3$, so we can see that

$$d(x, y) = d(x_1, x_2) = 3i \succ 2i = i + i = d(x_1, x_3) + d(x_3, x_2) = d(x, z) + d(z, y).$$

So d is not a complex valued metric.

Now define a mapping $T : X \rightarrow X$ as follows:

$$T(x) = \begin{cases} x_1, & \text{if } x = x_4 \\ x_3, & \text{if } x \in \{x_1, x_2, x_3\} \end{cases}$$

Note that

$$d(Tx_1, Tx_2) = d(Tx_1, Tx_3) = d(Tx_2, Tx_3) = 0$$

Also, in all other cases

$$d(Tx, Ty) = i, [d(x, Tx) + d(y, Ty)] \succ 5i.$$

Therefore, for $\lambda = \frac{1}{2}$, all conditions of Theorem 2.2 are satisfied to get a unique fixed point x_3 of T .

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